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Sampling per mode for rare event simulation in switching diffusions[☆]

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Abstract

A straightforward application of an interacting particle system to estimate a rare event for switching diffusions fails to produce reasonable estimates within a reasonable amount of simulation time. To overcome this, a conditional "sampling per mode" algorithm has been proposed by Krystul in [10]; instead of starting the algorithm with particles randomly distributed, we draw in each mode, a fixed number particles and at each resampling step, the same number of particles is sampled for each visited mode. In this paper, we establish a law of large numbers as well as a central limit theorem for the estimate.

MSC: 60F05; 60C05

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1. Introduction

Rare event simulation requires acceleration techniques to speed up the occurrence of the rare events under consideration, otherwise it may take unacceptably large sample sizes to get enough

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positive realizations, or even a single one, on average. A well known technique is *importance sampling*, whose idea is to change the probability laws driving the model in order to make the events of interest more likely, and to correct the bias by multiplying the estimator by the suitable likelihood ratio [13]. An alternative technique called *multilevel splitting* does not need to modify the probability laws that drive the system; this means that the computer program that implements the simulation model can just be a black box [12]. The idea of the splitting is to express the small probability of a rare event to be estimated as the product of a certain number of larger probabilities, which can be efficiently estimated by the Monte Carlo methods. This can be achieved by introducing sets of intermediate states that are visited by the stochastic process one after the other, in an ordered sequence, before reaching the final set of rare event states. The probability of a rare event is then given by the product of the conditional probabilities of reaching a set of intermediate states given that the previous set of intermediate states have been reached. Each conditional probability is estimated by simulating in parallel several copies of the system, i.e. each copy is considered as a particle following the trajectory generated through the system dynamics. Each particle branches (i.e. the trajectory splits into a number of independent subpaths, which subsequently evolve independently of each other) as soon as it enters the intermediate states, which is usually characterized by the crossing of a threshold defined by an *importance function*. Reaching intermediate states is more likely than reaching the rare event states, and by splitting at each threshold the chances to reach the rare event states are increasing. Several strategies have been designed to determine the importance function, to decide the number of splits at each level, and to handle the trajectories that tend to go in the wrong direction (away from the rare event of interest). It is most difficult to find an appropriate importance function; a poor choice can easily lead to bad results [7,8,11]. One of the best-known versions of splitting is the RESTART method [16–18].

The multilevel splitting technique can also be considered as an interacting particle interpretation (IPS) of the Feynman–Kac models, a general framework presented in [5]. This abstract Feynman–Kac formulation gives a powerful tool which in particular allows one to establish a strong law of large numbers and a central limit theorem for the estimate of the rare event probability [5,6,4].

Owing to the increasing demands for modelling large-scale and complex systems, switching diffusions (a subclass of hybrid processes) are lately receiving growing attention [1,10,19,2]. A distinctive feature of these systems is the coexistence of continuous dynamics and discrete events; they also satisfy the strong Markov property. While in theory the IPS approach is virtually applicable to any strong Markov process, in practice the straightforward application of this approach to switching diffusions fails to produce reasonable estimates within a reasonable amount of simulation time. The reason is that there may be few if no particles in modes with small probabilities (i.e. “light” modes). This happens because each resampling step tends to sample more “heavy” particles from modes with higher probabilities, thus, “light” particles in the “light” modes tend to be discarded. By increasing the number of particles the IPS estimates should improve but only at the cost of substantially increased simulation time which makes the performance of IPS approach in switching diffusions similar to one of the standard Monte Carlo. To avoid this, a conditional “sampling per mode” algorithm has been proposed in [10]; instead of starting the algorithm with N particles randomly distributed, we draw in each mode j, a fixed number N^j particles and at each resampling step, the same number of particles is sampled for each visited mode. Using the techniques introduced in [5,14], we recently established a law of large numbers as well as a central limit theorem for the estimate of the rare event probability as the number of particles increase to infinity.

The rest of the paper is arranged as follows. In Section 2, we introduce the abstract of Feynman–Kac and particle theory in the context of multilevel splitting and we adapt this framework to take into account the discrete modes of the switching diffusion. In particular, we detail the conditional "sampling per mode" algorithm. Section 3 is devoted to the asymptotic behaviour of algorithm as the number of particles tends to infinity. Using an approach based on a martingale decomposition, as presented in [5], we establish a law of large numbers and a central limit theorem for the particle approximation of the rare event probability. More precisely, let us denote respectively by γ and γ^N the probability of the rare event and its estimator, then we show that the estimator is unbiased and that it satisfies

$$\mathbb{E}\left([\gamma^N - \gamma]^2\right) \leq \frac{c}{N_{\text{inf}}},$$

where c is a constant which depends only on the number of intermediate states introduced in the algorithm and N_{inf} is the infimum of the number of particles N^j assigned to each mode. Furthermore, we obtain a central limit theorem which states that: if each N^j tends to infinity in such a way that the ratio N^j/N converges to a positive constant ρ_j, then $\sqrt{N}(\gamma^N - \gamma)$ converges in law to a Gaussian random variable with mean 0 and variance W given by $W = \gamma^2 \sum_j \rho_j^{-1} W_j^2$, where each W_j^2 is a variance term which depends on the mode j in some sense. This expression is similar in form to the expression of the variance obtained in [15] for the stratified sampling method. Thus, in order to make the estimation of γ more accurate than with a classical IPS algorithm, it takes to have a relatively homogeneity of the probability of achieving the rare event in each mode, hence one needs a suitable choice of the N^j. Thereby, the expression of the asymptotic variance can be used to get an idea of how best to implement the algorithm, following for instance the approach given in [11]. With this in mind, we do not seek to establish the most general result, but only that which provides us the asymptotic variance of the estimator of the rare event. We would also like to mention not to have established the explicit form of the variance of the estimator as it was made for the classic algorithm in [3]. It would of course be preferable to obtain such an expression, but it would take to extend the methods introduced in [3] to weighted empirical measures. Finally, Section 4 summarizes the paper and outlines a number of directions for future research.

2. Multilevel Feynman–Kac distributions

2.1. Formulation

We consider a switching diffusion $Z = \{(X_t, \theta_t); t \geq 0\}$ taking values in $\mathbb{R}^d \times \mathbb{M}$ for some $d > 0$ and $\mathbb{M} = \{1, \ldots, M\}$. More precisely, Z_t is a two-component Markov process such that (X_t) is a continuous component taking values in $\mathbb{R}^d$ and (θ_t) is a jump process taking values in the finite set $\mathbb{M}$; it can be described by

$$dX_t = b\,(X_t, \theta_t)\,dt + \sigma\,(X_t, \theta_t)\,dB_t,$$

and

$$\mathbb{P}\left(\theta_{t+\Delta t} = j | \theta_t = i, X_t = x\right) = \lambda_{ij}(x)\Delta t + o(\Delta t), \quad i \neq j$$

where B_t is a d-dimensional standard Brownian motion, $b(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{M} \to \mathbb{R}^n$ and $\sigma(\cdot, \cdot) : \mathbb{R}^n \times \mathbb{M} \to \mathbb{R}^{n \times d}$.

Throughout the paper, we assume that Z is a strong Markov process and we denote by η_0 the law of Z_0. Let $D \subset \mathbb{R}^d$ be some closed critical region, in which the continuous component of Z could enter but with a very low probability. Let us introduce an embedded sequence of closed regions

$$[0, \infty) \times D = D_n \subset \cdots \subset D_1 \subset D_0 \subset [0, \infty) \times \mathbb{R}^d,$$

which are usually defined by

$$D_k = \{(t, x) \in [0, \infty) \times \mathbb{R}^d : h(t, x) \leq c_k\},$$

where h is some lower semi-continuous function, called the *importance function*, and the cylinders $B = D \times \mathbb{M}$ and $A_k = D_k \times \mathbb{M}$. Set the corresponding hitting times

$$T_k = \inf\{t \geq 0 : (t, Z_t) \in A_k\} = \inf\{t \geq 0 : (t, X_t) \in D_k\},$$

which satisfy

$$0 = T_0 \leq T_1 \leq \cdots \leq T_n = T_B.$$

To capture the precise behaviour of the process Z in each region, we consider the random excursions $\mathcal{Z}_k$ of Z between the successive random times T_{k-1} and T_k. More precisely, we introduce the discrete-time Markov chain $\mathcal{Z} = \{\mathcal{Z}_k, k = 1, \ldots, n\}$ with values in the excursion set E, defined by

$$\mathcal{Z}_k = ((t, X_t, \theta_t), T_{k-1} \wedge T \leq t \leq T_k \wedge T),$$

with $t \wedge T = \inf\{T, t\}$ and T a deterministic or stopping time. We observe that these excursions can be decomposed in a string of diffusions, one by discrete mode; each of them having a random length. The non-homogeneous Markov kernels $\mathcal{M}_k$, which describe the Markovian transitions of the Markov chain $\mathcal{Z}$, are defined for all excursions e and all functions f on E by

$$\mathcal{M}_k f(e) = \mathbb{E}\left[f(\mathcal{Z}_k) | \mathcal{Z}_{k-1} = e\right].$$

To check whether or not a given path $e = ((t, Z_t), t_1 \leq t \leq t_2)$, starting at $(t_1, Z_{t_1}) \in A_{k-1}$ at time t_1, has succeeded to reach the level A_k at time t_2, it is convenient to introduce the terminal point $\pi(e) = (t_2, Z_{t_2})$ of the excursion and the indicator functions g_k defined by

$$g_k(e) = 1_{\{\pi(e) \in A_k\}},$$

and to capture the discrete component, we introduce the potential functions

$$g_k^j(e) = 1_{\{\pi(e) \in D_k \times \{j\}\}}, \quad j \in \mathbb{M},$$

giving the following decomposition

$$g_k(e) = \sum_{j \in \mathbb{M}} g_k^j(e). \tag{2.1}$$

With these notations, and for each k, we have $T_k \leq T$ if and only if $g_k(\mathcal{Z}_k) = 1$, and we have $T_k \leq T$ with $\theta_{T_k} = j$ if and only if $g_k^j(\mathcal{Z}_k) = 1$, hence

$$1_{\{T_k \leq T\}} = g_k(\mathcal{Z}_k), \quad \text{and} \quad 1_{\{T_k \leq T, \theta_{T_k} = j\}} = g_k^j(\mathcal{Z}_k).$$

Now define for $k = 1, \ldots, n$ the so-called unnormalized Feynman–Kac measures, γ_k and $\widehat{\gamma}_k$ on the path space E in such a way that the integral of all bounded measurable functions f relatively to these measures are given by

$$\begin{aligned}
\gamma_k(f) &= \mathbb{E}\left[f(\mathcal{Z}_k)g_{k-1}(\mathcal{Z}_{k-1})\right] \\
&= \mathbb{E}\left[f((t, X_t, \theta_t), T_{k-1} \le t \le T_k \wedge T)1_{\{T_{k-1} \le T\}}\right], \\
\widehat{\gamma}_k(f) &= \mathbb{E}\left[f(\mathcal{Z}_k)g_k(\mathcal{Z}_k)\right] \\
&= \mathbb{E}\left[f((t, X_t, \theta_t), T_{k-1} \le t \le T_k)1_{\{T_k \le T\}}\right].
\end{aligned}$$

In particular, when f is the constant function 1, we obtain

$$\gamma_k(1) = \mathbb{P}[T_{k-1} \le T], \quad \text{and} \quad \widehat{\gamma}_k(1) = \mathbb{P}[T_k \le T].$$

The Feynman–Kac distributions η_k and $\widehat{\eta}_k$ are derived by normalizing these measures,

$$\begin{aligned}
\eta_k(f) &= \frac{\gamma_k(f)}{\gamma_k(1)} = \mathbb{E}\left[f((t, X_t, \theta_t), T_{k-1} \le t \le T_k \wedge T)|T_{k-1} \le T\right], \\
\widehat{\eta}_k(f) &= \frac{\widehat{\gamma}_k(f)}{\widehat{\gamma}_k(1)} = \mathbb{E}\left[f((t, X_t, \theta_t), T_{k-1} \le t \le T_k)|T_k \le T\right],
\end{aligned}$$

and using the convention $T_{-1} = 0$ leads to the observation that $\gamma_0 = \eta_0$.

We observe that

$$1_{\{T_k \le T\}} = 1_{\{T_{k-1} \le T, T_k \le T\}},$$

or equivalently

$$g_k(\mathcal{Z}_k) = g_{k-1}(\mathcal{Z}_{k-1})g_k(\mathcal{Z}_k),$$

hence

$$\gamma_k(fg_k) = \widehat{\gamma}_k(f), \quad \text{and} \quad \widehat{\eta}_k(f) = \frac{\gamma_k(fg_k)}{\gamma_k(g_k)} = \frac{\eta_k(fg_k)}{\eta_k(g_k)}, \tag{2.2}$$

and more interestingly, how the probabilities of transition from one region to the following are related to the Feynman–Kac distributions,

$$\begin{aligned}
\eta_k(g_k) &= \mathbb{P}[T_k \le T|T_{k-1} \le T] := P_k, \\
\eta_k(g_k^j) &= \mathbb{P}[T_k \le T, \theta_{T_k} = j|T_{k-1} \le T] := P_k^j.
\end{aligned}$$

From now on, we assume that not only $P_k \ne 0$ for all k, but also that P_k^j are non zero for all modes $j \in \mathbb{M}$. Furthermore, the "unnormalized models" $(\gamma_k, \widehat{\gamma}_k)$ are related to the Feynman–Kac distribution flow $(\eta_p)_{p \le k}$, by the following key formula [5, Proposition 2.3.1]

$$\gamma_k(f) = \eta_k(f)\prod_{p=0}^{k-1}\eta_p(g_p) \quad \text{and} \quad \widehat{\gamma}_k(f) = \widehat{\eta}_k(f)\prod_{p=0}^{k}\eta_p(g_p). \tag{2.3}$$

In order to keep trace of the discrete mode, we construct for any $j \in \mathbb{M}$ the unnormalized Feynman–Kac measures γ_k^j and $\widehat{\gamma}_k^j$ defined by

$$\begin{aligned}
\gamma_k^j(f) &= \mathbb{E}\left[f(\mathcal{Z}_k)g_{k-1}^j(\mathcal{Z}_{k-1})\right] \\
&= \mathbb{E}\left[f((t, X_t, \theta_t), T_{k-1} \leq t \leq T_k \wedge T) 1_{\{T_{k-1} \leq T, \theta_{T_{k-1}} = j\}}\right], \\
\widehat{\gamma}_k^j(f) &= \mathbb{E}\left[f(\mathcal{Z}_k)g_k^j(\mathcal{Z}_k)\right] \\
&= \mathbb{E}\left[f((t, X_t, \theta_t), T_{k-1} \leq t \leq T_k) 1_{\{T_k \leq T, \theta_{T_k} = j\}}\right].
\end{aligned}$$

The Feynman–Kac distributions η_k^j and $\widehat{\eta}_k^j$ are derived by normalizing these measures, respectively

$$\begin{aligned}
\eta_k^j(f) &= \frac{\gamma_k^j(f)}{\gamma_k^j(1)} = \mathbb{E}\left[f((t, X_t, \theta_t), T_{k-1} \leq t \leq T_k \wedge T) | T_{k-1} \leq T, \theta_{T_{k-1}} = j\right], \\
\widehat{\eta}_k^j(f) &= \frac{\widehat{\gamma}_k^j(f)}{\widehat{\gamma}_k^j(1)} = \mathbb{E}\left[f((t, X_t, \theta_t), T_{k-1} \leq t \leq T_k) | T_k \leq T, \theta_{T_k} = j\right].
\end{aligned}$$

We observe that

$$1_{\{T_k \leq T, \theta_{T_k} = j\}} = 1_{\{T_{k-1} \leq T, T_k \leq T, \theta_{T_k} = j\}},$$

or equivalently

$$g_k^j(\mathcal{Z}_k) = g_{k-1}(\mathcal{Z}_{k-1})g_k^j(\mathcal{Z}_k),$$

hence

$$\gamma_k(fg_k^j) = \widehat{\gamma}_k^j(f), \quad \text{and} \quad \widehat{\eta}_k^j(f) = \frac{\gamma_k(fg_k^j)}{\gamma_k(g_k^j)} = \frac{\eta_k(fg_k^j)}{\eta_k(g_k^j)}. \tag{2.4}$$

Clearly, we have the decompositions

$$\widehat{\eta}_k = \sum_{j \in \mathbb{M}} \omega_k^j \widehat{\eta}_k^j, \qquad \eta_{k+1} = \sum_{j \in \mathbb{M}} \omega_k^j \eta_{k+1}^j \tag{2.5}$$

where

$$\omega_k^j = \widehat{\eta}_k(g_k^j) = \frac{\eta_k(g_k^j)}{\eta_k(g_k)} = \frac{\widehat{\gamma}_k^j(1)}{\widehat{\gamma}_k(1)} = \mathbb{P}(\theta_{T_k} = j | T_k \leq T). \tag{2.6}$$

2.2. Feynman–Kac semigroups

Previously, we have introduced the Feynman–Kac distributions η_k. Now, we will investigate the time evolution of the Feynman–Kac flow $(\eta_k; 0 \leq k \leq n)$. In fact, from the Markov property of the process Z, we see that

$$\gamma_k(f) = \mathbb{E}\left[g_{k-1}(\mathcal{Z}_{k-1})\mathbb{E}\left[f(\mathcal{Z}_k)|\mathcal{Z}_{k-1}\right]\right] \tag{2.7}$$

$$= \gamma_{k-1}\left(g_{k-1}\mathcal{M}_k f\right) = \widehat{\gamma}_{k-1}\left(\mathcal{M}_k f\right), \tag{2.8}$$

and in the context of switching jump diffusions, for any $j \in \mathbb{M}$

$$\gamma_k^j(f) = \mathbb{E}\left[g_{k-1}^j(\mathcal{Z}_{k-1})\mathbb{E}\left[f(\mathcal{Z}_k)|\mathcal{Z}_{k-1}\right]\right] \tag{2.9}$$

$$= \gamma_{k-1}\left(g_{k-1}^j \mathcal{M}_k f\right) = \widehat{\gamma}_{k-1}^j\left(\mathcal{M}_k f\right). \tag{2.10}$$

These formulae suggest the introduction of the linear operators Q_k and Q_k^j defined respectively by

$$Q_k f = g_{k-1}\mathcal{M}_k f \quad \text{and} \quad Q_k^j f = g_{k-1}^j \mathcal{M}_k f. \tag{2.11}$$

An immediate consequence of (2.7) and (2.9) is that γ_k and γ_k^j satisfy linear equations of the form

$$\gamma_k = \gamma_{k-1} Q_k \quad \text{and} \quad \gamma_k^j = \gamma_{k-1} Q_k^j. \tag{2.12}$$

Nevertheless, we seek the evolution of the normalized Feynman–Kac measures η_k and η_k^j, that we suspect to be nonlinear. To establish it, we introduce the mappings Φ_k from the set of measures $\mathcal{P}_k(E) = \{\eta \colon \eta(g_k) > 0\}$ into the set $\mathcal{P}(E)$ of measure on E defined by

$$\Phi_k(\eta)(f) = \Psi_{k-1}(\eta)\left(\mathcal{M}_k f\right), \quad \text{with } \Psi_k(\eta)(f) = \frac{\eta(f g_k)}{\eta(g_k)}.$$

It can now be easily verified that the Feynman–Kac flow is the solution of a nonlinear measure-valued dynamical system

$$\eta_k = \Phi_k(\eta_{k-1}). \tag{2.13}$$

Since $\Psi_k(\eta_k) = \widehat{\eta}_k$, we see that the recursion (2.13) involves two separate selection/mutation transitions

$$\eta_k \in \mathcal{P}_k(E) \xrightarrow{\text{selection}} \widehat{\eta}_k := \Psi_k(\eta_k) \in \mathcal{P}(E) \xrightarrow{\text{mutation}} \eta_{k+1} = \widehat{\eta}_k \mathcal{M}_{k+1} \in \mathcal{P}(E). \tag{2.14}$$

In the specific case of switching jump diffusions, we introduce for any $j \in \mathbb{M}$ the following transformations

$$\Phi_k^j(\eta)(f) := \Psi_{k-1}^j(\eta)\left(\mathcal{M}_k f\right) \quad \text{with } \Psi_{k-1}^j(\eta)(f) := \frac{\eta\left(g_{k-1}^j f\right)}{\eta\left(g_{k-1}^j\right)},$$

and from (2.4), we check that

$$\widehat{\eta}_k^j = \Psi_k^j(\eta_k), \quad \text{and} \quad \eta_{k+1}^j = \Phi_{k+1}^j(\eta_k) = \widehat{\eta}_k^j \mathcal{M}_{k+1}. \tag{2.15}$$

Furthermore, the operator Φ_k can be written as the weighted sum

$$\Phi_k(\eta) = \sum_{j \in \mathbb{M}} \frac{\eta(g_{k-1}^j)}{\eta(g_{k-1})} \Phi_k^j(\eta), \tag{2.16}$$

and from (2.6), we deduce the particular case

$$\Phi_k(\eta_{k-1}) = \sum_{j \in \mathbb{M}} \omega_{k-1}^j \Phi_k^j(\eta_{k-1}).$$

We have seen how the measure η_n is related to the previous measure η_{n-1}, but we will need to express η_n in terms of η_p for all $0 \le p \le n$. Nevertheless, the evolution being nonlinear it is easier to start by considering the evolution of γ_n and γ_n^j, since a straightforward iteration of (2.12) leads us to introduce for any $j \in \mathbb{M}$, the linear semigroups $Q_{p,n}$ and $Q^j_{p,n}$, defined respectively by

$$Q_{p,n} = Q_{p+1} Q_{p+2} \cdots Q_n, \qquad Q^j_{p,n} = Q_{p+1} Q_{p+2} \cdots Q_{n-1} Q^j_n = Q_{p,n-1} Q^j_n, \quad (2.17)$$

with the convention $Q_{n,n} = \mathrm{Id}$. We remark first that

$$Q_{p,n}\, 1 = Q_{p,n-1}\, g_{n-1}, \qquad Q^j_{p,n}\, 1 = Q_{p,n-1}\, g^j_{n-1}.$$

Second, we get

$$\eta_n(f) = \frac{\gamma_p\left(Q_{p,n}\, f\right)}{\gamma_p\left(Q_{p,n}\, 1\right)} = \frac{\eta_p\left(Q_{p,n}\, f\right)}{\eta_p\left(Q_{p,n}\, 1\right)}, \quad (2.18)$$

and, for any $j \in \mathbb{M}$

$$\gamma^j_n(f) = \gamma_p\left(Q^j_{p,n}\, f\right), \qquad \eta^j_n(f) = \frac{\eta_p\left(Q^j_{p,n}\, f\right)}{\eta_p\left(Q^j_{p,n}\, 1\right)}. \quad (2.19)$$

Note that an immediate consequence of (2.18) and (2.19) is that

$$\eta_p\left(Q_{p,n}\, 1\right) = \frac{\gamma_n(1)}{\gamma_p(1)} = \prod_{q=p}^{n-1} \eta_q(g_q), \quad \text{and} \quad \eta_p\left(Q^j_{p,n}\, 1\right) = \frac{\gamma^j_n(1)}{\gamma_p(1)}. \quad (2.20)$$

We obtain also

$$\omega^j_{n-1} = \frac{\eta_p\left(Q^j_{p,n}\, 1\right)}{\eta_p\left(Q_{p,n}\, 1\right)}. \quad (2.21)$$

2.3. *Interacting particle system approximations*

2.3.1. *Introduction*

From a pure mathematical point of view, particle methods can be interpreted as a kind of stochastic linearization technique for solving nonlinear equations in measure space. The idea is to associate to the nonlinear dynamical structure (2.13) a sequence $\xi = (\xi^1, \ldots, \xi^N)$, of N excursion-valued particles, such that the empirical measure of the configurations converge as $N \to \infty$ to the desired distribution η.

More precisely, at time k, we consider the particles

$$\xi^i_k = \Big((t, X^i_t, \theta^i_t), ;\, T^i_{k-1} \le t \le T^i_k \wedge T\Big) \in E \cup \{\Delta\},$$
$$\widehat{\xi}^i_k = \Big((t, \widehat{X}^i_t, \widehat{\theta}^i_t), ;\, \widehat{T}^i_{k-1} \le t \le \widehat{T}^i_k\Big) \in E \cup \{\Delta\},$$

where Δ stands for a cemetery point that we introduce here to take into account the possible stopping of the algorithm. The random lengths of the corresponding excursions ξ^i_k and $\widehat{\xi}^i_k$ are $T^i_k \wedge T - T^i_{k-1}$ and $\widehat{T}^i_k - \widehat{T}^i_{k-1}$ respectively.

The interacting particle system (IPS) approach consists in approximating the two step transitions (2.14) of the system (2.13) by the two step transitions

$$\eta_k^N := \frac{1}{N}\sum_{i=1}^N \delta_{\xi_k^i} \xrightarrow{\text{selection}} \widehat{\eta}_k^N := \frac{1}{N}\sum_{i=1}^N \delta_{\widehat{\xi}_k^i} \xrightarrow{\text{mutation}} \eta_{k+1} = \frac{1}{N}\sum_{i=1}^N \delta_{\xi_{k+1}^i}. \tag{2.22}$$

So, starting from an approximation η_0^N to η_0, during the mutation transition $\widehat{\xi}_k \to \xi_{k+1}$, each selected particle $\widehat{\xi}_k^\kappa$ evolves randomly according to the Markov transition $\mathcal{M}_{k+1}$, independently of each other. The selection transition $\xi_{k+1} \to \widehat{\xi}_{k+1}$ is defined as follows: only some of the particles ξ_{k+1} have succeeded in reaching the desired set A_{k+1}; unless the amount of these is equal to zero (in this case, the algorithm is stopped), we sample, randomly and independently, the N particles $\widehat{\xi}_{k+1}$ distributed according to the distribution $\Psi_{k+1}(\eta_{k+1}^N)$.

The IPS is nothing else that a sequence of nonhomogeneous Markov chains on the product space E^N with transition kernels given by

$$\mathbb{P}(\widehat{\xi}_k \in dx|\xi_k) = \prod_{i=1}^N \Psi_k(\eta_k^N)(dx^i)$$

$$\mathbb{P}(\xi_k \in dz|\widehat{\xi}_{k-1}) = \prod_{i=1}^N \mathcal{M}_k(\widehat{\xi}_{k-1}^i, dz^i),$$

and initial measures $\eta_0^N = \widehat{\eta}_0^N$.

Nevertheless, the classical IPS algorithm is not really suitable for switching jump diffusion, mainly because of the potential existence of discrete modes with very small initial probability. To avoid the particles never been drawn in these "light" modes, a conditional "sampling per mode" algorithm has been proposed in [10]; instead of starting the algorithm with N_0 particles randomly distributed in $A_0 \setminus A_1$ according to η_0, we draw in each mode j, a fixed number N^j particles in $(D_0 \setminus D_1) \times \{j\}$, randomly distributed according to the conditional law η_0^j. Resampling per mode allows us to avoid loss of "light" particles in "light modes", and so helps us to maintain a fixed number of particles in each mode. We will see hereafter, that the empirical measure η_k^N is a weighted sum of the conditional empirical measure $\eta_k^{j,N}$ (see (2.5))

$$\eta_k^N = \sum_j \omega_{k-1}^{j,N}\eta_k^{j,N} = \sum_\kappa \beta_k^\kappa \delta_{\xi_k^\kappa}, \quad \text{with } \eta_k^{j,N} = \frac{1}{N^j}\sum_\kappa \delta_{\xi_k^\kappa},$$

where the three sums are taken over, respectively, all discrete modes containing at least one particle, all particles and finally all particles in the discrete mode j. Nevertheless, the total number of particles can decrease, typically if one or several modes become empty at some time, and it can also increase, typically if one or several of these empty modes become non empty at a later time. That means we have to introduce the total number N_k of particles, at each step k.

2.3.2. Detailed description

Prior to the detailed description of the IPS, it is convenient to introduce some notations. Let J_k be the set of non empty discrete modes after the end of step k, with the convention $J_0 = \mathbb{M}$, and for each $j \in J_k$, we set

$$J_k^j = \{\kappa : \pi(\xi_k^\kappa) \in D_k \times \{j\}\},$$
$$\widehat{J}_k^j = \{\kappa : \pi(\widehat{\xi}_k^\kappa) \in D_k \times \{j\}\},$$
$$\widehat{I}_k^N = \bigcup_{j \in J_k} \widehat{J}_k^j.$$

Specifically, J_k^j is the set of indices of particles ξ_k^κ that have reached the set A_k in mode j, before the final time T, while $\widehat{J}_k^j$ is the set of indices of selected particles $\widehat{\xi}_k^\kappa$ whose terminal point is in the discrete mode j, and finally $\widehat{I}_k^N$ is the set of indices of all selected particles $\widehat{\xi}_k^\kappa$.

As the total number of particles can decrease as soon as none of the particles have succeeded to reach the desired level in some mode, we need to introduce the following sequence of integers

$$\widehat{N}_k = |\widehat{I}_k^N| = \sum_{j \in J_k} N^j = \sum_{j \in \mathbb{M}} \widehat{N}_k^j,$$

where $\widehat{N}_k^j = N^j$ if $j \in J_k$ and $\widehat{N}_k^j = 0$ otherwise. Thus, our model is not restricted to a fixed population size but it takes values in the state space

$$\mathbf{E} = \bigcup_{p \in \mathbb{N}} (\{p\} \times [0, \infty)^p \times E^p)$$

with the convention $E^0 = \{\Delta\}$; the parameter p represents the size of the system. It is advisable to observe that we can have $\widehat{N}_{k+1}^j = N^j$ while $\widehat{N}_k^j = 0$; consequently the size of the population can increase or decrease by jumps.

We associate with the time evolution of this model

$$(N_k, \beta_k, \xi_k) \xrightarrow{\text{selection}} (\widehat{N}_k, \widehat{\beta}_k, \widehat{\xi}_k) \xrightarrow{\text{mutation}} (N_{k+1}, \beta_{k+1}, \xi_{k+1}) \tag{2.23}$$

the canonical filtrations $F_k \subset \widehat{F}_k \subset F_{k+1}$. Then the IPS algorithm is conducted inductively as follows:

Initialization: For each $j \in J_0$, we sample N^j particles $\xi_0^\kappa = \widehat{\xi}_0^\kappa = (0, (X_0^\kappa, j)) \sim \eta_0^j$, with $\kappa \in J_0^j$. How the label κ is ascribed to each particle is explained in detail in Section 3.2. Let $\omega_0^j = \mathbb{P}(\theta_0 = j)$, then the empirical measures of particles η_0^N and $\widehat{\eta}_0^N$ are given by

$$\eta_0^N = \sum_{j \in \mathbb{M}} \omega_0^j \eta_0^{j,N} = \sum_{\kappa=1}^{N_0} \beta_0^\kappa \delta_{\xi_0^\kappa}, \quad \text{with } \eta_0^{j,N} = \frac{1}{N^j} \sum_{\kappa \in J_0^j} \delta_{\xi_0^\kappa},$$

$$\widehat{\eta}_0^N = \sum_{j \in \mathbb{M}} \omega_0^j \widehat{\eta}_0^{j,N} = \sum_{\kappa=1}^{N_0} \widehat{\beta}_0^\kappa \delta_{\widehat{\xi}_0^\kappa}, \quad \text{with } \widehat{\eta}_0^{j,N} = \frac{1}{N^j} \sum_{\kappa \in \widehat{J}_0^j} \delta_{\widehat{\xi}_0^\kappa},$$

where $\beta_0^\kappa = \widehat{\beta}_0^\kappa = \omega_0^j / N^j$, for each $\kappa \in J_0^j = \widehat{J}_0^j$.

At each step k, the empirical measure $\widehat{\eta}_k^N$ will be given by

$$\widehat{\eta}_k^N = \sum_{\kappa \in \widehat{I}_k^N} \widehat{\beta}_k^\kappa \delta_{\widehat{\xi}_k^\kappa} = \sum_{j \in J_k} \omega_k^{j,N} \widehat{\eta}_k^{j,N}, \quad \text{where } \widehat{\eta}_k^{j,N} = \frac{1}{N^j} \sum_{\kappa \in \widehat{J}_k^j} \delta_{\widehat{\xi}_k^\kappa}, \tag{2.24}$$

and the weights $\omega_k^{j,N}$ are nonnegative and such that $\sum_{j\in J_k}\omega_k^{j,N}=1$. Clearly, $\widehat{\beta}_k^\kappa=\omega_k^{j,N}/N^j$ for each $\kappa\in\widehat{J}_k^j$. We notice that the particles in the same discrete mode have the same weight, or in other words, the weight of a particle depends only on the mode, i.e. on its discrete component.

The *mutation transition* $\widehat{\xi}_k\to\xi_{k+1}$ at time $k+1$ is defined as follows. If $\widehat{N}_k=0$, the particle system dies and we set $N_{k+1}=0$. Otherwise during mutation, independently of each other, each selected particle $\widehat{\xi}_k^\kappa$ evolves randomly according to the Markov transition $\mathcal{M}_{k+1}$; in other words,

$$\xi_{k+1}^\kappa=\big((t,X_t^\kappa,\theta_t^\kappa),T_k^\kappa\le t\le T_{k+1}^\kappa\wedge T\big),$$

is a random variable with distribution $\mathcal{M}_{k+1}(\widehat{\xi}_k^\kappa,\cdot)$. More precisely, we set $T_k^\kappa=\widehat{T}_k^\kappa$ and the path $\big((t,X_t^\kappa,\theta_t^\kappa),t\ge T_k^\kappa\big)$ advances randomly as a copy of the process $\big((t,X_t,\theta_t),t\ge T_k^\kappa\big)$, i.e. according to the dynamics of the switching diffusion, starting at $(T_k^\kappa,\widehat{X}_{T_k^\kappa},\widehat{\theta}_{T_k^\kappa})$ and up to the first time T_{k+1}^κ it visits A_{k+1}, or up to T, whichever occurs first. During this transition, the total number of particles does not change, so we set $N_{k+1}=\widehat{N}_k$.

The weight of each particle is set as $\beta_{k+1}^\kappa=\widehat{\beta}_k^\kappa$ and the N-particle approximation measure is given by

$$\eta_{k+1}^N=\sum_{\kappa\in\widehat{I}_k^N}\beta_{k+1}^\kappa\delta_{\xi_{k+1}^\kappa}=\sum_{j\in J_k}\omega_k^{j,N}\eta_{k+1}^{j,N},$$

where

$$\eta_{k+1}^{j,N}=\frac{1}{N^j}\sum_{\kappa\in\widehat{J}_k^j}\delta_{\xi_{k+1}^\kappa}.$$

We easily verify that

$$\mathbb{E}\left[\eta_{k+1}^{j,N}(f)\middle|\widehat{F}_k\right]=\widehat{\eta}_k^{j,N}(\mathcal{M}_{k+1}f),\qquad \mathbb{E}\left[\eta_{k+1}^{N}(f)\middle|\widehat{F}_k\right]=\widehat{\eta}_k^{N}(\mathcal{M}_{k+1}f). \tag{2.25}$$

The *selection transition* $\xi_{k+1}\to\widehat{\xi}_{k+1}$ is defined as follows. From the N_{k+1} particles ξ_{k+1}^κ, only some of them have succeeded to reach the desired set A_{k+1}; we recall that

$$J_{k+1}^j=\{\kappa:\pi(\xi_{k+1}^\kappa)\in D_{k+1}\times\{j\}\}$$

and we set

$$I_{k+1}^N=\bigcup_{j\in J_k}J_{k+1}^j.$$

If $I_{k+1}^N=\emptyset$, then none of the particles have succeeded to reach the desired region; the algorithm is stopped and $\widehat{\xi}_{k+1}=\Delta$. Otherwise, it may happen that there are some j for which $J_{k+1}^j=\emptyset$, but as long as I_{k+1}^N is not empty, we still continue the algorithm.

An empty mode j, such that $J_{k+1}^j=\emptyset$, remains empty, i.e. no particle is sampled in this mode so that $\widehat{J}_{k+1}^j=\emptyset$. For each non empty mode j, such that $J_{k+1}^j\neq\emptyset$, we need to sample N^j particles in this mode. Nevertheless, more or less than N^j particles may have reached the set $D_{k+1}\times\{j\}$, so we need to resample N^j particles among the particles ξ_{k+1}^κ, with $\kappa\in J_{k+1}^j$. If $|J_{k+1}^j|\le N^j$, then not enough particles have managed to reach the set $D_{k+1}\times\{j\}$ before the final time T, and these successful particles should be replicated, whereas if $|J_{k+1}^j|>N^j$, then too many particles have managed to reach the set $D_{k+1} \times \{j\}$ before the final time T, and some of these successful particles should be eliminated.

More precisely, we choose randomly the N^j particles $\widehat{\xi}^\kappa_{k+1}$, $\kappa \in \widehat{J}^j_{k+1}$, identically distributed according to the distribution

$$\Psi^j_{k+1}(\eta^N_{k+1}) = \sum_{\kappa \in J^j_{k+1}} \left(\frac{\beta^\kappa_{k+1}}{\sum\limits_{\kappa \in J^j_{k+1}} \beta^\kappa_{k+1}} \right) \delta_{\xi^\kappa_{k+1}}.$$

By construction $\widehat{\xi}^\kappa_{k+1}$ is the copy of a successful particle ξ^τ_{k+1} with $\tau \in J^j_{k+1}$; necessarily $\widehat{T}^\kappa_{k+1} = T^\tau_{k+1} \leq T$. Each particle ξ^κ_{k+1} for $\kappa \in J^j_{k+1}$ branches into a random number of offsprings $M^{j,\kappa}_{k+1}$ and the sequence $(M^{j,\kappa}_{k+1}, \kappa \in J^j_{k+1})$ is distributed according to a

$$(M^{j,\kappa}_{k+1}, \kappa \in J^j_{k+1}) = \text{Multinomial} \left(N^j, \left(\frac{\beta^\kappa_{k+1}}{\sum\limits_{\kappa \in J^j_{k+1}} \beta^\kappa_{k+1}}, \kappa \in J^j_{k+1} \right) \right).$$

For each $j \in J_{k+1}$, we obtain

$$\widehat{\eta}^{j,N}_{k+1} = \frac{1}{N^j} \sum_{\kappa \in \widehat{J}^j_{k+1}} \delta_{\widehat{\xi}^\kappa_{k+1}} = \frac{1}{N^j} \sum_{\kappa \in J^j_{k+1}} M^{j,\kappa}_{k+1} \delta_{\xi^\kappa_{k+1}};$$

namely, we approximate the empirical measure $\Psi^j_{k+1}(\eta^N_{k+1})$ by a new probability measure whose atom weights are integer multiples of $1/N^j$.

Furthermore, the mechanism is such that for any bounded test function f, we have

$$\mathbb{E}\left(\widehat{\eta}^{j,N}_{k+1}(f) | F_{k+1} \right) = \Psi^j_{k+1}(\eta^N_{k+1})(f), \tag{2.26}$$

and

$$\begin{aligned} \mathbb{E}\left(\left[\widehat{\eta}^{j,N}_{k+1}(f) - \Psi^j_{k+1}(\eta^N_{k+1})(f) \right]^2 \middle| F_{k+1} \right) &= \frac{1}{N^j} \operatorname{var}\left(f, \Psi^j_{k+1}(\eta^N_{k+1}) \right) \\ &\leq \|f\|^2 / N^j, \end{aligned} \tag{2.27}$$

where $\operatorname{var}(f, \mu) = \mu[(f - \mu(f))^2]$ and $\|f\| = \sup_{x \in E} |f(x)|$.

The total number $\widehat{N}_{k+1}$ of particles $\widehat{\xi}^\kappa_{k+1}$ and the weights $\omega^{j,N}_{k+1}$ are given respectively by

$$\widehat{N}_{k+1} = \sum_{j \in J_{k+1}} N^j, \quad \text{and} \quad \omega^{j,N}_{k+1} = \frac{\eta^N_{k+1}(g^j_{k+1})}{\eta^N_{k+1}(g_{k+1})}.$$

This would define $\widehat{\eta}^N_{k+1}$ by the use of Eq. (2.24). Now, it follows from

$$\frac{\eta^N_{k+1}(g^j_{k+1})}{\eta^N_{k+1}(g_{k+1})} = \frac{\sum\limits_{\kappa \in J^j_{k+1}} \beta^\kappa_{k+1}}{\sum\limits_{\kappa \in I^N_{k+1}} \beta^\kappa_{k+1}},$$

that

$$\omega_{k+1}^{j,N} = \sum_{\kappa \in J_{k+1}^{j}} \left(\frac{\beta_{k+1}^{\kappa}}{\sum_{\kappa \in I_{k+1}^{N}} \beta_{k+1}^{\kappa}} \right),$$

and $\widehat{\beta}_{k+1}^{\kappa} = \omega_{k+1}^{j,N}/N^{j}$ for any $\kappa \in \widehat{J}_{k+1}^{j}$.

Using (2.3) and (2.6), we also introduce the measures γ_n^N and $\gamma_n^{j,N}$ defined respectively by

$$\gamma_n^N(f) = \eta_n^N(f) \prod_{p=0}^{n-1} \eta_p^N(g_p) \quad \text{and} \quad \gamma_n^{j,N}(f) = \omega_{n-1}^{j,N} \gamma_n^N(1) \eta_n^{j,N}(f),$$

as approximations of γ_n and γ_n^j. In particular for $f \equiv 1$, the IPS algorithm provides

$$\gamma_{n+1}^N(1) = \prod_{p=0}^{n} \eta_p^N(g_p) = \prod_{p=0}^{n} \sum_{\kappa \in I_p^N} \beta_p^{\kappa},$$

as an estimate of the rare event probability $\mathbb{P}(T_n \leq T) = \gamma_{n+1}(1)$. In other words, $\gamma_{n+1}^N(1)$ is the product of proportions of excursions having entered levels $A_1, \ldots, A_n$.

3. Asymptotic behaviour

In this section, our aim is to examine the asymptotic behaviour of particle approximation models as the number of particles tends to infinity. We start with the analysis of the unnormalized measure γ_n^N and we show that this approximation has no bias. We follow the approach used in [5] which is based on a martingale decomposition. Finally, we establish a central limit theorem for unnormalized particle approximation measures.

3.1. Law of large numbers

We begin by introducing some useful formulae and inequalities. First, on the event $\{N_k > 0\}$, we have

$$\mathbb{E}(\widehat{\eta}_k^N(f)|F_k) = \sum_{j \in J_k} \mathbb{E}(\omega_k^{j,N} \widehat{\eta}_k^{j,N}(f)|F_k) = \sum_{j \in J_k} \omega_k^{j,N} \Psi_k^j(\eta_k^N)(f) = \Psi_k(\eta_k^N)(f), \tag{3.1}$$

since $\omega_k^{j,N} = \eta_k^N(g_k^j)/\eta_k^N(g_k)$ is measurable w.r.t. F_k. Introducing the integer $N_{\inf}$ defined by $N_{\inf} = \inf\{N^j : j = 1, \ldots, M\}$, we deduce that

$$\begin{aligned} \mathbb{E}\left[\left(\widehat{\eta}_k^N(f) - \Psi_k(\eta_k^N)(f)\right)^2 \Big| F_k\right] &= \sum_{j \in J^k} \frac{(\omega_k^{j,N})^2}{N^j} \operatorname{var}(f, \Psi_k^j(\eta_k^N)) \\ &\leq \|f\|^2 \sum_{j \in J_k} \frac{(\omega_k^{j,N})^2}{N^j} \leq N_{\inf}^{-1} \|f\|^2. \end{aligned} \tag{3.2}$$

Second, when $J_k \neq \emptyset$ we obtain by (2.25) that

$$\begin{aligned}
\mathbb{E}\left(\eta_{k+1}^N(f)\middle| F_k, \widehat{F}_k\right) &= \sum_{j\in J_k} \omega_k^{j,N} \mathbb{E}\left(\widehat{\eta}_k^{j,N}\,(\mathcal{M}_{k+1}\,f)\,\middle| F_k\right) \\
&= \sum_{j\in J_k} \omega_k^{j,N}\, \Psi_k^j(\eta_k^N)\,(\mathcal{M}_{k+1}\,f) \\
&= \sum_{j\in J_k} \omega_k^{j,N}\, \Phi_{k+1}^j(\eta_k^N)(f) \\
&= \sum_{j\in J_k} \frac{\eta_k^N(g_k^j)}{\eta_k^N(g_k)}\, \Phi_{k+1}^j(\eta_k^N)(f) \\
&= \Phi_{k+1}(\eta_k^N)(f),
\end{aligned}$$

where we used (2.16) for the last line. It follows easily that

$$\mathbb{E}\left(\eta_{k+1}^N(f)|F_k\right) = \Phi_{k+1}(\eta_k^N)(f). \tag{3.3}$$

We now need to compute the conditional variance of $\eta_{k+1}^{j,N}(f)$, i.e.

$$\mathrm{Var}\left(\eta_{k+1}^{j,N}(f)|F_k\right) := \mathbb{E}\left(\left[\eta_{k+1}^{j,N}(f) - \Phi_{k+1}^j(\eta_k^N)(f)\right]^2 \middle| F_k\right).$$

Recall that for any random variable X and any σ-algebra $\mathcal{F}$, we have the following relation

$$\mathrm{Var}(X) = \mathrm{Var}\left(\mathbb{E}(X|\mathcal{F})\right) + \mathbb{E}\left(\mathrm{Var}(X|\mathcal{F})\right),$$

so that

$$\mathrm{Var}\left(\eta_{k+1}^{j,N}(f)|F_k\right) = \mathrm{Var}\left(\mathbb{E}(\eta_{k+1}^{j,N}(f)|\widehat{F}_k)|F_k\right) + \mathbb{E}\left(\mathrm{Var}(\eta_{k+1}^{j,N}(f)|\widehat{F}_k)|F_k\right).$$

As the particles ξ_{k+1}^κ are independent conditionally to $\widehat{F}_k$, we get by (2.25) that

$$\mathrm{Var}(\eta_{k+1}^{j,N}(f)|\widehat{F}_k) = \frac{1}{N^j}\widehat{\eta}_k^{j,N}\left[\mathcal{M}_{k+1}(f^2) - (\mathcal{M}_{k+1}f)^2\right],$$

and hence by (2.26)

$$\begin{aligned}
\mathrm{Var}\left(\eta_{k+1}^{j,N}(f)|F_k\right) &= \mathrm{Var}\left(\widehat{\eta}_k^{j,N}(\mathcal{M}_{k+1}f)|F_k\right) \\
&\quad + \frac{1}{N^j}\,\Psi_k^j(\eta_k^N)\left[\mathcal{M}_{k+1}(f^2) - (\mathcal{M}_{k+1}f)^2\right].
\end{aligned}$$

Now, using (2.27) gives

$$\begin{aligned}
\mathrm{Var}\left(\eta_{k+1}^{j,N}(f)|F_k\right) &= \frac{1}{N^j}\mathrm{var}(\mathcal{M}_{k+1}f,\, \Psi_k^j(\eta_k^N)) \\
&\quad + \frac{1}{N^j}\,\Psi_k^j(\eta_k^N)\left[\mathcal{M}_{k+1}(f^2) - (\mathcal{M}_{k+1}f)^2\right] \\
&= \frac{1}{N^j}\mathrm{var}(f,\, \Phi_{k+1}^j(\eta_k^N)).
\end{aligned}$$

Hence, we deduce the following

$$\mathbb{E}\left(\left[\eta_{k+1}^N(f) - \Phi_{k+1}(\eta_k^N)(f)\right]^2 \Big| F_k\right) = \sum_{j\in J_k} \frac{(\omega_k^{j,N})^2}{N^j} \mathrm{var}\left(f, \Phi_{k+1}^j(\eta_k^N)\right) \le \frac{\|f\|^2}{N_{\mathrm{inf}}}. \quad (3.4)$$

Now, we study the difference between the particle measure γ_n^N and the limiting Feynman–Kac measures γ_n. Following the approach given in [5], we use the decomposition for each bounded function f

$$\begin{aligned} 1_{\{N_n>0\}}\gamma_n^N(f) - \gamma_n(f) &= \left[\gamma_0^N(Q_{0,n} f) - \gamma_0(Q_{0,n} f)\right] \\ &+ \sum_{p=1}^n \left[1_{\{N_p>0\}}\gamma_p^N(Q_{p,n} f) - 1_{\{N_{p-1}>0\}}\gamma_{p-1}^N(Q_{p-1,n} f)\right]. \end{aligned} \quad (3.5)$$

In other respects, since

$$1_{\{N_p>0\}} = 1_{\{N_{p-1}>0,N_p>0\}} = 1_{\{N_{p-1}>0\}} - 1_{\{N_{p-1}>0,N_p=0\}},$$

and

$$1_{\{N_{p-1}>0,N_p=0\}}\eta_p^N = 0,$$

we conclude that

$$\begin{aligned} 1_{\{N_{p-1}>0\}}\gamma_{p-1}^N(Q_{p-1,n} f) &= 1_{\{N_{p-1}>0\}}\gamma_{p-1}^N(1)\eta_{p-1}^N\left(Q_p(Q_{p,n} f)\right) \\ &= 1_{\{N_{p-1}>0\}}\gamma_p^N(1)\,\Phi_p(\eta_{p-1}^N)(Q_{p,n} f), \end{aligned}$$

and

$$\begin{aligned} 1_{\{N_p>0\}}\gamma_p^N(Q_{p,n} f) &= 1_{\{N_p>0\}}\eta_p^N(Q_{p,n} f)\gamma_p^N(1) \\ &= 1_{\{N_{p-1}>0\}}\gamma_p^N(1)\eta_p^N(Q_{p,n} f). \end{aligned}$$

This yields the formula

$$\begin{aligned} &1_{\{N_p>0\}}\gamma_p^N(Q_{p,n} f) - 1_{\{N_{p-1}>0\}}\gamma_{p-1}^N(Q_{p-1,n} f) \\ &= 1_{\{N_{p-1}>0\}}\gamma_p^N(1)\left[\eta_p^N(Q_{p,n} f) - \Phi_p(\eta_{p-1}^N)(Q_{p,n} f)\right]. \end{aligned} \quad (3.6)$$

On taking expectation and using (3.3) we thus deduce that

$$\mathbb{E}[\gamma_p^N(Q_{p,n} f) - \gamma_{p-1}^N(Q_{p-1,n} f)|F_{p-1}] = 0.$$

Proposition 1. *For any $n \ge 0$ and bounded function f, the $\mathbb{R}$-valued process $\Gamma_{\cdot,n}(f)$ defined by*

$$\Gamma_{p,n}(f) := 1_{\{N_p>0\}}\gamma_p^N(Q_{p,n} f) - \gamma_p(Q_{p,n} f), \quad p \le n,$$

is an F-martingale with increasing process given by the formula

$$\begin{aligned} \langle\Gamma_{\cdot,n}(f)\rangle_p &= \mathbb{E}\left(\left[\eta_0^N(Q_{0,n} f) - \eta_0(Q_{0,n} f)\right]^2\right) \\ &+ \sum_{q=1}^p 1_{\{N_{q-1}>0\}}\left(\gamma_q^N(1)\right)^2 \mathbb{E}\left(\left[\eta_q^N(Q_{q,n} f) - \Phi_q(\eta_{q-1}^N)(Q_{q,n} f)\right]^2 |F_{q-1}\right). \end{aligned} \quad (3.7)$$

Proof. For all functions φ, we deduce from (3.5) and (3.6) that

$$1_{\{N_p>0\}}\gamma_p^N(\varphi) - \gamma_p(\varphi) = \left[\eta_0^N(Q_{0,p}\,\varphi) - \eta_0(Q_{0,p}\,\varphi)\right] + \sum_{q=1}^{p} \gamma_q^N(1) 1_{\{N_{q-1}>0\}} \left[\eta_q^N(Q_{q,p}\,\varphi) - \Phi_q(\eta_{q-1}^N)(Q_{q,p}\,\varphi)\right].$$

Therefore by choosing $\varphi = Q_{p,n}\, f$, we get

$$\Gamma_{p,n}(f) = \left[\eta_0^N(Q_{0,n}\, f) - \eta_0(Q_{0,n}\, f)\right] + \sum_{q=1}^{p} 1_{\{N_{q-1}>0\}}\gamma_q^N(1) \left[\eta_q^N(Q_{q,n}\, f) - \Phi_q(\eta_{q-1}^N)(Q_{q,n}\, f)\right], \tag{3.8}$$

and (3.7) is a clear consequence of the above decomposition. □

Corollary 2. *For any $n \geq 0$ and any bounded function f, we have*

$$\mathbb{E}(\gamma_n^N(f) 1_{\{N_n>0\}}) = \gamma_n(f),$$

and

$$\sup_{f:\,\|f\|\leq 1} \mathbb{E}\left([1_{\{N_n>0\}}\gamma_n^N(f) - \gamma_n(f)]^2\right) \leq \frac{b^2(n)}{N_{\inf}},$$

for some finite constant $b(n) = c(n+1)$.

Proof. The first assertion follows from the martingale property of $(\Gamma_{p,n}(f))_p$ and the second from the martingale property of $(\Gamma_{p,n}^2(f) - \langle \Gamma_{\cdot,n}(f)\rangle_p)_p$ and the observation that in view of (3.4)

$$\mathbb{E}\left(\langle \Gamma_{\cdot,n}(f)\rangle_n\right) \leq (n+1)\frac{C}{N_{\inf}}\|f\|^2 := \frac{b^2(n)}{N_{\inf}}\|f\|^2. \quad \square$$

This corollary shows that the IPS approximation γ_n^N of the unnormalized Feynman–Kac measure γ_n has zero bias and mean-square error (or variance) of order $1/N_{\inf}$. The end of this section is devoted to showing that similar results hold for the IPS approximation η_n^N of the normalized Feynman–Kac distribution η_n as well: the bias is of order $1/N_{\inf}$ and the mean square error is also of order $1/N_{\inf}$. The second statement follows readily from the decomposition

$$1_{\{N_n>0\}}\eta_n^N(f) - \eta_n(f) = \frac{1_{\{N_n>0\}}\gamma_n^N(f) - \gamma_n(f)}{\gamma_n(1)} - 1_{\{N_n>0\}}\eta_n^N(f)\frac{1_{\{N_n>0\}}\gamma_n^N(1) - \gamma_n(1)}{\gamma_n(1)},$$

which implies by using the triangle inequality

$$\left[\mathbb{E}\left|1_{\{N_n>0\}}\eta_n^N(f) - \eta_n(f)\right|^2\right]^{1/2} \leq \left[\mathbb{E}\left|\frac{1_{\{N_n>0\}}\gamma_n^N(f) - \gamma_n(f)}{\gamma_n(1)}\right|^2\right]^{1/2} + \|f\|\left[\mathbb{E}\left|\frac{1_{\{N_n>0\}}\gamma_n^N(1) - \gamma_n(1)}{\gamma_n(1)}\right|^2\right]^{1/2},$$

and hence

$$\sup_{f:\,\|f\|\le 1}\left[\mathbb{E}\left|1_{\{N_n>0\}}\eta_n^N(f)-\eta_n(f)\right|^2\right]^{1/2}\le\frac{2b^2(n)}{(\gamma_n(1))^2N_{\inf}}.$$

Now to address the estimation of the bias, we will argue as in the proof of Theorem 7.4.3 in [5]. We start with the following decomposition

$$\left(\eta_n^N(f)-\eta_n(f)\right)1_{\{N_n>0\}}=\frac{\gamma_n(1)}{\gamma_n^N(1)}\gamma_n^N(f_n)1_{\{N_n>0\}},\tag{3.9}$$

where $f_n=\frac{1}{\gamma_n(1)}(f-\eta_n(f))$. Since, $\gamma_n(f_n)=0$, (3.9) also reads

$$\left(\eta_n^N(f)-\eta_n(f)\right)1_{\{N_n>0\}}=\frac{\gamma_n(1)}{\gamma_n^N(1)}\left(\gamma_n^N(f_n)1_{\{N_n>0\}}-\gamma_n(f_n)\right)1_{\{N_n>0\}}.$$

By Corollary 2, $\mathbb{E}\left(\gamma_n^N(f_n)1_{\{N_n>0\}}\right)=\gamma_n(f_n)=0$, this implies that

$$0=\mathbb{E}\left[\gamma_n^N(f_n)1_{\{N_n>0\}}-\gamma_n(f_n)\right]=\mathbb{E}\left[\left(\gamma_n^N(f_n)1_{\{N_n>0\}}-\gamma_n(f_n)\right)1_{\{N_n>0\}}\right]$$

and finally, we get the formula

$$\begin{aligned}&\mathbb{E}\left[\left(\eta_n^N(f)-\eta_n(f)\right)1_{\{N_n>0\}}\right]\\&\quad=\mathbb{E}\left[\frac{\gamma_n(1)}{\gamma_n^N(1)}\left(1-\frac{\gamma_n^N(1)}{\gamma_n(1)}\right)1_{\{N_n>0\}}\left(\gamma_n^N(f_n)1_{\{N_n>0\}}-\gamma_n(f_n)\right)\right].\end{aligned}$$

Now we set $h_n=\frac{1}{\gamma_n(1)}$ and conclude

$$\begin{aligned}\mathbb{E}\left[\left(\eta_n^N(f)-\eta_n(f)\right)1_{\{N_n>0\}}\right]&=-\mathbb{E}\left[\frac{\gamma_n(1)}{\gamma_n^N(1)}\left(\gamma_n^N(h_n)1_{\{N_n>0\}}-\gamma_n(h_n)\right)\right.\\&\qquad\left.\times\left(\gamma_n^N(f_n)1_{\{N_n>0\}}-\gamma_n(f_n)\right)1_{\{N_n>0\}}\right].\end{aligned}\tag{3.10}$$

In order to control the term $\gamma_n(1)/\gamma_n^N(1)$, we consider the set Ω_n^N of events

$$\Omega_n^N=\{\gamma_n^N(1)1_{\{N_n>0\}}\ge\gamma_n(1)/2\}=\left\{\frac{\gamma_n(1)}{\gamma_n^N(1)}\le 2\text{ and }N_n>0\right\}\subset\{N_n>0\}.$$

Moreover, combining Corollary 2 with Chebyshev's inequality gives the following inequality

$$\mathbb{P}\left(\gamma_n^N(1)1_{\{N_n>0\}}\ge\gamma_n(1)/2\right)\ge 1-\frac{4b^2(n)}{(\gamma_n(1))^2N_{\inf}},\tag{3.11}$$

from which we get directly that

$$\mathbb{P}\left((\Omega_n^N)^c\right)\le\frac{4b^2(n)}{(\gamma_n(1))^2N_{\inf}}.$$

Putting things together yields that for any f with $\|f\| \leq 1$

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(\eta_n^N(f) - \eta_n(f)\right) 1_{\{N_n>0\}}\right]\right| \\
&\quad \leq \left|\mathbb{E}\left[\left(\eta_n^N(f) - \eta_n(f)\right) 1_{\Omega_n^N}\right]\right| + \mathbb{E}\left[(|\eta_n^N(f)| + |\eta_n(f)|) 1_{(\Omega_n^N)^c}\right] \\
&\quad \leq \left|\mathbb{E}\left[\left(\eta_n^N(f) - \eta_n(f)\right) 1_{\Omega_n^N}\right]\right| + 2\mathbb{P}\left((\Omega_n^N)^c\right) \\
&\quad \leq 2\mathbb{E}\left(\left|\gamma_n^N(h_n) 1_{\{N_n>0\}} - \gamma_n(h_n)\right| \left|\gamma_n^N(f_n) 1_{\{N_n>0\}} - \gamma_n(f_n)\right|\right) + \frac{8b^2(n)}{(\gamma_n(1))^2 N_{\text{inf}}}.
\end{aligned}$$

We combine now the Corollary 2 and the Cauchy–Schwarz inequality in order to obtain

$$\left|\mathbb{E}\left[\left(\eta_n^N(f) - \eta_n(f)\right) 1_{\{N_n>0\}}\right]\right| \leq \frac{12b^2(n)}{(\gamma_n(1))^2 N_{\text{inf}}},$$

since $\|f_n\| \leq 2/\gamma_n(1)$ and $\|h_n\| \leq 1/\gamma_n(1)$, and finally, we get

$$\left|\mathbb{E}\left[\eta_n^N(f) 1_{\{N_n>0\}} - \eta_n(f)\right]\right| \leq \frac{12b^2(n)}{(\gamma_n(1))^2 N_{\text{inf}}} + \mathbb{P}(N_n = 0).$$

Now, we need to estimate the extinction probability of the algorithm. In [5, Theorem 7.4.1], a tricky proof gives the following bound for the extinction time τ^N of the general particle algorithm

$$\mathbb{P}(\tau^N \leq n) \leq a(n) e^{-N/c(n)}, \tag{3.12}$$

for some constants $a(n)$ and $c(n)$. Nevertheless, we can apply this bound with N_{inf} instead of N (see the Appendix for a detailed proof). Hence, as $\mathbb{P}(N_n = 0) = \mathbb{P}(\tau^N < n)$, we have obtained the following result.

Theorem 3. *For each $n \in \mathbb{N}$ and for any bounded function f, we have*

$$\sup_{f:\, \|f\| \leq 1} \left|\mathbb{E}\left[\eta_n^N(f) 1_{\{N_n>0\}} - \eta_n(f)\right]\right| \leq \frac{b^2(n)}{N_{\text{inf}}} + a(n) e^{-N_{\text{inf}}/c^2(n)},$$

and

$$\sup_{f:\, \|f\| \leq 1} \left[\mathbb{E}\left|1_{\{N_n>0\}} \eta_n^N(f) - \eta_n(f)\right|^2\right]^{1/2} \leq \frac{b^2(n)}{N_{\text{inf}}}$$

for some finite constants $a(n)$, $c(n) = c \times (n+1)/\gamma_n(1)$ and $b(n) = b \times (n+1)/\gamma_n(1)$.

Remark 1. Applying this Theorem with $f = g^j \varphi$ where φ is a function such that $\|\varphi\| \leq 1$, we obtain the same inequalities for the measure $1_{\{|J_{n-1}^j|>0\}} \eta_n^{j,N}$ instead of $\eta_n^N(f) 1_{\{N_n>0\}}$. For instance,

$$\sup_{f:\, \|f\| \leq 1} \left[\mathbb{E}\left|1_{\{|J_{n-1}^j|>0\}} \eta_n^{j,N}(f) - \eta_n^j(f)\right|^2\right]^{1/2} \leq \frac{b^2(n)}{N_{\text{inf}}}.$$

3.2. Central limit theorem

We now tackle the Central Limit Theorem (CLT) whose importance for stochastic algorithms is well known, mainly to obtain a confidence interval for the estimator, and more importantly the expression of the asymptotic variance that can be used to get an idea of how best to implement the algorithm. For instance, how to size the number of particles in each mode to obtain the smallest asymptotic variance. With this in mind, we do not seek to establish the most general result, but only that which provides us the asymptotic variance of the estimator of the rare event. We will follow the approach based on the CLT for triangular arrays introduced in [5]. Nevertheless, as the number of particles is random, we will need to adapt this approach according to [14].

In order to introduce a triangular array of random variables, we split up the terminal term of the martingale $(\Gamma_{\cdot,n}(f))$ as a sum over the particles, so we obtain by (3.8)

$$\sqrt{N}\Gamma_{n,n}(f) = \sum_{q=1}^{n}\sum_{\kappa=1}^{N_q}\sqrt{N}\beta_q^{\kappa}1_{\{N_{q-1}>0\}}\gamma_q^N(1)\left[f_q(\xi_q^{\kappa}) - \Phi_q(\eta_{q-1}^N)(f_q)\right] + \sqrt{N}\left[\eta_0^N(f_0) - \eta_0(f_0)\right],$$

with $f_q := Q_{q,n} f$ for $0 \le q \le n$. Introducing the random variables

$$X_{q,\kappa}^N := \sqrt{N}\beta_q^{\kappa}1_{\{N_{q-1}>0\}}\gamma_q^N(1)\left[f_q(\xi_q^{\kappa}) - \Phi_q(\eta_{q-1}^N)(f_q)\right], \quad 1 \le q \le n,$$
$$X_{0,\kappa}^N := \sqrt{N}\beta_0^{\kappa}[f_0(\xi_0^{\kappa}) - \eta_0(f_0)],$$

we can rewrite $\sqrt{N}\Gamma_{n,n}(f)$ in the following form

$$\sqrt{N}\Gamma_{n,n}(f) = \sum_{q=0}^{n}\sum_{\kappa=1}^{N_q} X_{q,\kappa}^N.$$

Furthermore, we can write

$$\sqrt{N}\Gamma_{n,n}(f) = \sum_{j\in\mathbb{M}}\sqrt{N/N^j}\sqrt{N^j}\Gamma_{n,n}^j(f),$$

with

$$\sqrt{N^j}\Gamma_{n,n}^j(f) = \frac{1}{\sqrt{N^j}}\sum_{\kappa\in J_0^j}\omega_0^j\left[f_0(\xi_0^{\kappa}) - \eta_0^j(f_0)\right] + \frac{1}{\sqrt{N^j}}\sum_{q=1}^{n}\sum_{\kappa\in\widehat{J}_{q-1}^j}1_{\{|J_{q-1}^j|>0\}}\gamma_q^{j,N}(1)\left[f_q(\xi_q^{\kappa}) - \Phi_q^j(\eta_{q-1}^N)(f_q)\right].$$

We recall that if $|J_{q-1}^j| > 0$, then $|\widehat{J}_{q-1}^j| = N^j$. So, introducing the random variables

$$X_{q,\kappa}^{j,N} := \frac{1}{\sqrt{N^j}}1_{\{|J_{q-1}^j|>0\}}\gamma_q^{j,N}(1)\left[f_q(\xi_q^{\kappa}) - \Phi_q^j(\eta_{q-1}^N)(f_q)\right], \quad 1 \le q \le n,$$
$$X_{0,\kappa}^{j,N} := \frac{1}{\sqrt{N^j}}\omega_0^j\left[f_0(\xi_0^{\kappa}) - \eta_0^j(f_0)\right],$$

we obtain the following form

$$\sqrt{N^j}\Gamma_{n,n}^j(f) = \sum_{q=1}^{n}\sum_{\kappa\in\widehat{J}_{q-1}^j}X_{q,\kappa}^{j,N} + \sum_{\kappa\in J_0^j}X_{0,\kappa}^{j,N}.$$

Finally, let us mention that $X_{q,\kappa}^N = \sqrt{N/N^j}X_{q,\kappa}^{j,N}$ for $\kappa \in \widehat{J}_{q-1}^j$.

Henceforth, at the end of each selection/resampling step, the indices of each particle will be ordered according to the order induced by the modes. More precisely, setting $N_q = \sum_{j\in\mathbb{M}} N_q^j$ with $N_q^j = N^j$ if $j \in J_{q-1}$ and $N_q^j = 0$ otherwise, we label the $N_q = \widehat{N}_{q-1}$ particles $(\widehat{\xi}^{\kappa}_{q-1})$ in such a way that $\kappa \in \widehat{J}^j_{q-1}$ if and only if the interval

$$I_j := \left[\sum_{i=1}^{j-1} N_q^i + 1, \sum_{i=1}^{j} N_q^i\right] \quad \text{with } N_q^0 := 0$$

is not empty and $\kappa \in I_j$. We remind the reader that for $\kappa \in \widehat{J}^j_{q-1}$ a particle ξ_q^κ is an excursion started in $D_{q-1} \times \{j\}$ with $j \in J_{q-1}$. Now, let us introduce some new notations: first, we set

$$K_q^N = N_0 + \cdots + N_q, \quad 0 \le q \le n,$$

with the convention $K_{-1}^N = 0$. Second, we subdivide each interval $[K_{q-1}^N + 1, K_q^N]$ in a reunion of the mutually disjoint sub-intervals $[K_{q,j-1}^N + 1, K_{q,j}^N]$ $(j \in \mathbb{M})$, where

$$K_{q,j}^N = K_{q-1}^N + N_q^1 + \cdots + N_q^j, \qquad K_{q,0}^N = K_q^N, \quad \text{and} \quad K_{q,M}^N = K_{q+1}^N.$$

Obviously, for any $j \notin J_{q-1}$, such an interval is empty.

We notice that the κ-th particle within the q-th generation can be associated in a unique way with an integer k between 1 and K_n^N and a mode $j \in J_{q-1}$: clearly $k_{q,\kappa} = K_{q-1}^N + \kappa$ and $j = \inf\{i : K_{q,i}^N \ge k\}$. Conversely, for a fixed integer $1 \le k \le K_n^N$, the random integers q_k, κ_k and j_k are defined by

$$q_k = \inf\{q \ge 0 : K_q^N \ge k\} \qquad \kappa_k = k - K_{q_k-1}^N \quad \text{and} \quad j_k = \inf\{j : K_{q_k,j}^N \ge k\},$$

or in other words $q_k = q$ and $\kappa_k = \kappa$ if and only if

$$K_{q-1}^N + 1 \le k = K_{q-1}^N + \kappa \le K_q^N.$$

We also check that the set of particles ξ_q^κ such that $\kappa \in \widehat{J}^j_{q-1}$ for any $j \in J_{q-1}$ is the set of particles whose index k belongs to the interval $\left[K_{q,j-1}^N + 1, K_{q,j}^N\right]$ which is non empty as soon as $j \in J_{q-1}$.

Now, we introduce a filtration $\mathcal{G}^N = \{\mathcal{G}_k^N, k \ge 1\}$ in such a way that K_n^N is a stopping time w.r.t. $\mathcal{G}^N$. For any $q = 0, 1, \ldots, n$ and any integer $\kappa \ge 1$, let $\mathcal{F}_{q,\kappa}^N = \mathcal{F}_{q,0}^N \vee \sigma(\xi_q^1, \ldots, \xi_q^\kappa)$, where $\mathcal{F}_{0,0}^N = \{\emptyset, \Omega\}$ and $\mathcal{F}_{q,0}^N = F_{q-1}$ (for $q = 1, \ldots, n$). Since $\{N_q = p\} \in \mathcal{F}_{q,p}^N$ by definition, the random number N_q is a stopping time w.r.t. the filtration $\mathcal{F}_q^N = \{\mathcal{F}_{q,\kappa}^N, \kappa \ge 0\}$, which allows to define the σ-algebra $\mathcal{F}_{q,N_q}^N := F_q$. Therefore, the random variable K_q^N is measurable w.r.t. F_q.

For any $q = 0, 1, \ldots, n$ and any integer $\kappa \ge 1$,

$$\{q_k = q, \kappa_k = \kappa\} = \{k = K_{q-1}^N + \kappa, \kappa \le N_q\} \subset \mathcal{F}_{q,\kappa-1}^N \subset \mathcal{F}_{q,\kappa}^N,$$

since $\{k = K_{q-1}^N + \kappa\} \in F_{q-1}$ and $\{\kappa \le N_q\} \in \mathcal{F}_{q,\kappa-1}^N$. Then, we can define in the usual way the σ-algebra $\mathcal{G}_k^N = \mathcal{F}_{q_k,\kappa_k}^N$ by: $A \in \mathcal{G}_k^N$ if and only if $A \cap \{q_k = q, \kappa_k = \kappa\} \in \mathcal{F}_{q,\kappa}^N$, for any

$q = 0, 1, \ldots, n$ and any integer $\kappa \geq 1$. Using this new labelling of the particle system yields

$$\sqrt{N}\Gamma_{n,n}(f) = \sum_{k=1}^{K_n^N} U_k^N,$$

where $U_k^N := X_{q_k,\kappa_k}^N$ is measurable w.r.t. $\mathcal{G}_k^N$, for any $k = 1, \ldots, K_n^N$: indeed for any Borel subset B

$$\{U_k^N \in B\} \cap \{q_k = q, \kappa_k = \kappa\} = \{X_{q,\kappa}^N \in B\} \cap \{q_k = q, \kappa_k = \kappa\},$$

hence $\{U_k^N \in B\} \in \mathcal{G}_k^N$, since $\{X_{q,\kappa}^N \in B\} \in \mathcal{F}_{q,\kappa}^N$ and $\{q_k = q, \kappa_k = \kappa\} \in \mathcal{F}_{q,\kappa-1}^N$.

Moreover, the random variable K_n^N is a stopping time w.r.t. $\mathcal{G}^N$, since

$$\begin{aligned}\{K_n^N = k\} \cap \{q_k = q, \kappa_k = \kappa\} &= \{K_n^N = k\} \cap \{k = K_{q-1}^N + \kappa, 1 \leq \kappa \leq N_q\}, \\ &= \begin{cases} \emptyset & \text{if } q \neq n, \\ \{k = K_{n-1}^N + \kappa\} \cap \{N_n = \kappa\} & \text{if } q = n, \end{cases}\end{aligned}$$

hence $\{K_n^N = k\} \in \mathcal{G}_k^N$ since $\{k = K_{n-1}^N + \kappa\} \in F_{n-1}$ and $\{N_n = \kappa\} \in \mathcal{F}_{n,\kappa}^N$. We also obtain the following expression

$$\begin{aligned}\sqrt{N}\Gamma_{n,n}(f) &= \sum_{j\in\mathbb{M}} \sum_{q=0}^{n} \sum_{k=K_{q,j-1}^N+1}^{K_{q,j}^N} U_k^N \\ &= \sum_{j\in\mathbb{M}} \sum_{q=0}^{n} \sum_{\kappa\in\widehat{J}_{q-1}^j} X_{q,\kappa}^N \\ &= \sum_{j\in\mathbb{M}} \sqrt{N/N^j} \sum_{q=0}^{n} \sum_{\kappa\in\widehat{J}_{q-1}^j} X_{q,\kappa}^{j,N}.\end{aligned}$$

Now, since $K_n^N < \infty$ a.s., to apply the Theorem VIII.3.33 in [9] we need to check the following three conditions

(i) $\mathbb{E}(U_k^N | \mathcal{G}_{k-1}^N) = 0$,
(ii) $\sum_{k=1}^{K_n^N} \mathbb{E}\left(|U_k^N|^2 | \mathcal{G}_{k-1}^N\right) \xrightarrow{P} \sigma^2$,
(iii) for all $\epsilon > 0$, $\sum_{k=1}^{K_n^N} \mathbb{E}\left(|U_k^N|^2 1_{\{|U_k^N|>\epsilon\}} | \mathcal{G}_{k-1}^N\right) \xrightarrow{P} 0$.

However, since $U_k^N = X_{q_k,\kappa_k}^N$ is a time changed random variable, we need the following result [14, Lemma 4, Corollaries 1 and 2].

Lemma 4. *If for any $q = 0, 1, \ldots, n$ and any integer $\kappa \geq 1$*

$$\mathbb{E}[F_{q,\kappa} | \mathcal{F}_{q,\kappa-1}^N] = \widehat{F}_q,$$

where the random variables $\widehat{F}_q$ is measurable w.r.t. $\mathcal{F}_{q,0}^N$, then for any integer $k \geq 1$, the time changed random variable $G_k = F_{q_k,\kappa_k}$ and $\widehat{G}_k = \widehat{F}_{q_k}$ satisfy

$$\mathbb{E}[G_k | \mathcal{G}_{k-1}^N] = \widehat{G}_k.$$

If for any $q = 0, 1, \dots, n$ and any integer $\kappa \geq 1$

$$F_{q,\kappa} \leq F_q^*,$$

where the random variable F_q^* is measurable w.r.t. $\mathcal{F}_{q,0}^N$, then for any integer $k \geq 1$, the time changed random variables $G_k = F_{q_k,\kappa_k}$ and $G_k^* = F_{q_k}^*$ satisfy

$$\mathbb{E}[G_k | \mathcal{G}_{k-1}^N] \leq G_k^*.$$

First step: For any $\kappa = 1, \dots, N_0$, the random variable $X_{0,\kappa}^N$ is measurable w.r.t. $\mathcal{F}_{0,\kappa}^N$. Moreover,

$$\mathbb{E}\left[X_{0,\kappa}^N | \mathcal{F}_{0,\kappa-1}^N\right] = 0,$$

and for any $\kappa \in J_0^j$

$$\mathbb{E}\left[|X_{0,\kappa}^{j,N}|^2 | \mathcal{F}_{0,\kappa-1}^N\right] = \frac{1}{N^j} (\omega_0^j)^2 \mathrm{var}\left(f_0; \eta_0^j\right) := V_{0,j}^N.$$

Notice that

$$|X_{0,\kappa}^{j,N}| \leq \frac{1}{\sqrt{N^j}} 2\|f_0\|,$$

hence for any $\epsilon > 0$

$$|X_{0,\kappa}^{j,N}|^2 1_{\{|X_{0,\kappa}^{j,N}| > \epsilon\}} \leq \frac{1}{N^j} (2\|f_0\|)^2 1_{\left\{\frac{1}{\sqrt{N^j}} 2\|f_0\| > \epsilon\right\}} := Y_{0,j}^N(\epsilon).$$

For any $q = 1, \dots, n$, and any $j \in J_{q-1}$, the random weight $\gamma_q^{j,N}(1)$ is measurable w.r.t. $F_{q-1} = \mathcal{F}_{q,0}^N$. Furthermore, for any $\kappa \in \widehat{J}_{q-1}^j$ the random variable $X_{q,\kappa}^{j,N}$ is measurable w.r.t. $\mathcal{F}_{q,\kappa}^N$, and conditionally w.r.t. F_{q-1}, the particles ξ_q^κ, with $\kappa \in \widehat{J}_{q-1}^j$, are i.i.d. with common distribution $\Phi_q^j(\eta_{q-1}^N)$. Moreover,

$$\mathbb{E}\left[X_{q,\kappa}^{j,N} | \mathcal{F}_{q,\kappa-1}^N\right] = 0, \tag{3.13}$$

and for $\kappa \in \widehat{J}_{q-1}^j$

$$\mathbb{E}[|X_{q,\kappa}^{j,N}|^2 | \mathcal{F}_{q,\kappa-1}^N] = \frac{1}{N^j} 1_{\{|J_{q-1}^j| > 0\}} \left(\gamma_q^{j,N}(1)\right)^2 \mathrm{var}(f_q, \Phi_q^j(\eta_{q-1}^N)) := V_{q,j}^N, \tag{3.14}$$

where the random variables $V_{q,j}^N$ are measurable w.r.t. $\mathcal{F}_{q,0}^N$. Notice that

$$|X_{q,\kappa}^{j,N}| \leq \frac{\gamma_q^{j,N}(1)}{\sqrt{N^j}} 2\|f_q\|,$$

hence for any $\epsilon > 0$,

$$|X_{q,\kappa}^{j,N}|^2 1_{\{|X_{q,\kappa}^{j,N}(f)| > \epsilon\}} \leq \frac{(\gamma_q^{j,N}(1))^2}{N^j} (2\|f_q\|)^2 1_{\left\{\frac{\gamma_q^{j,N}(1)}{\sqrt{N^j}} 2\|f_q\| > \epsilon\right\}} := Y_{q,j}^N(\epsilon), \tag{3.15}$$

where the random variable $Y^N_{q,j}(\epsilon)$ is measurable w.r.t. $\mathcal{F}^N_{q,0}$. It follows from (3.13) to (3.15) and Lemma 4 that

$$\mathbb{E}\left(U^N_k|\mathcal{G}^N_{k-1}\right) = 0$$
$$\mathbb{E}\left(|U^N_k|^2|\mathcal{G}^N_{k-1}\right) = (N/N^{j_k})V^N_{q_k,j_k}, \quad \text{if } \kappa_k \in J^{j_k}_{q_k-1},$$

and

$$\mathbb{E}\left(|U^N_k|^2 1_{\{|U^N_k|>\epsilon\}}|\mathcal{G}^N_{k-1}\right) \le (N/N^{j_k})Y^N_{q_k,j_k}\left(\epsilon\sqrt{N^{j_k}/N}\right),$$

hence,

$$\sum_{k=1}^{K^N_n}\mathbb{E}\left(|U^N_k|^2|\mathcal{G}^N_{k-1}\right) = \sum_{j\in\mathbb{M}} N V^N_{0,j} + \sum_{q=1}^{n}\sum_{j\in J_{q-1}} N V^N_{q,j}$$

and

$$\sum_{k=1}^{K^N_n}\mathbb{E}\left(|U^N_k|^2 1_{\{|U^N_k|>\epsilon\}}|\mathcal{G}^N_{k-1}\right) \le \sum_{j\in\mathbb{M}}\sum_{q=0}^{n} N Y^N_{q,j}\left(\epsilon\sqrt{N^j/N}\right).$$

Second step: We assume now that each N^j tends to infinity, in such a way that each ratio N^j/N converges to a positive constant ρ^j. Then, we deduce from Corollary 2 and Theorem 3 that $\gamma^{j,N}_q(1)1_{\{|J^j_{q-1}|>0\}} \xrightarrow{P} \gamma^j_q(1)$, and $\mathrm{var}\left(f_q;\, \Phi^j_q(\eta^N_{q-1})\right) \xrightarrow{P} \mathrm{var}\left(f_q;\, \Phi^j_q(\eta_{q-1})\right)$. Then, we get that

$$N V^N_{0,j} \xrightarrow{P} \frac{(\omega^j_0)^2}{\rho_j}\mathrm{var}\left(f_0;\, \eta^j_0\right),$$

and

$$N V^N_{q,j} \xrightarrow{P} \frac{(\gamma^j_q(1))^2}{\rho_j}\mathrm{var}\left(f_q;\, \eta^j_q\right).$$

Note also that $|J^j_{q-1}| > 0$ if and only if $\eta^N_{q-1}(g^j_{q-1}) > 0$, but this probability converges to P^j_{q-1} as N tends to infinity, so if we assume $P^j_{q-1} \neq 0$ (a reasonable assumption), we can conclude that

$$\sum_{k=1}^{K^N_n}\mathbb{E}\left(|U^N_k|^2|\mathcal{G}^N_{k-1}\right) \to W_n(f),$$

with

$$\begin{aligned} W_n(f) &= \sum_{j\in\mathbb{M}}\left(\sum_{q=1}^{n}\frac{(\gamma^j_q(1))^2}{\rho_j}\mathrm{var}(f_q,\eta^j_q) + \frac{(\omega^j_0)^2}{\rho_j}\mathrm{var}(f_0,\eta^j_0)\right) \\ &= \gamma_n(1)^2\sum_{j\in\mathbb{M}}\sum_{q=0}^{n}\frac{(\omega^j_{q-1})^2}{\rho_j}\frac{\mathrm{var}(f_q,\eta^j_q)}{\eta^2_q(Q_{q,n}1)}. \end{aligned}$$

To obtain the last line, we used the formula $\gamma_q^j(1) = \omega_{q-1}^j \gamma_n(1)/\eta_q(Q_{q,n}1)$ valid for $q \geq 1$ and the convention $\omega_{-1}^j = \omega_0^j$. Moreover, we can check that $\sum_{q=0}^n \sum_{j\in\mathbb{M}} NY_{q,j}^N(\epsilon\sqrt{N^j/N}) \to 0$ in probability. More precisely, each term of this sum has the form $Z_N 1_{\{Z_N > \epsilon\sqrt{N^j}\}}$ with $Z_N \xrightarrow{P} Z$ where Z is bounded. So, we have proved the CLT. □

Theorem 5. *Suppose each N^j tends to infinity in such a way that the ratio N^j/N converges to a positive constant ρ_j. Then, the sequence of random variables*

$$\sqrt{N}\left(1_{\{N_n>0\}}\gamma_n^N(g_n) - \mathbb{P}(T_n \leq T)\right)$$

converges in law to a Gaussian random variable with mean 0 and variance $W_n := W_n(g_n)$.

Remark 2. The asymptotic variance of the classical algorithm i.e. without sampling per mode is given by Cérou et al. [4]

$$V_n(f) = \gamma_n(1)^2 \sum_{q=0}^n \frac{\operatorname{var}(f_q, \eta_q)}{\eta_q^2(Q_{q,n}1)},$$

whereas, using (2.21)

$$W_n(f) = \gamma_n(1)^2 \sum_{j\in\mathbb{M}} \rho_j^{-1} \sum_{q=0}^n \frac{\operatorname{var}(f_q, \eta_q^j)}{\eta_q^2(Q_{q,n}^j 1)}.$$

But, we have

$$\begin{aligned}
\operatorname{var}(f_q, \eta_q) &= \sum_{j\in\mathbb{M}} \omega_{q-1}^j \operatorname{var}(f_q, \eta_q^j) + \sum_{j\in\mathbb{M}} \omega_{q-1}^j \left(\eta_q^j(f_q) - \eta_q(f_q)\right)^2 \\
&= \sum_{j\in\mathbb{M}} \omega_{q-1}^j \operatorname{var}(f_q, \eta_q^j) + \sum_{j\in\mathbb{M}} \omega_{q-1}^j (\eta_q^j(f_q))^2 - \eta_q^2(f_q) \\
&= \sum_{j\in\mathbb{M}} \omega_{q-1}^j \operatorname{var}(f_q, \eta_q^j) + \sum_{j\in\mathbb{M}} \omega_{q-1}^j [(\eta_q^j(f_q))^2 - \eta_q^2(f_q)],
\end{aligned}$$

then $V_n(f)$ can be written as

$$\frac{V_n(f)}{\gamma_n(1)^2} = \sum_{j\in\mathbb{M}} \sum_{q=0}^n \omega_{q-1}^j \frac{\operatorname{var}(f_q, \eta_q^j)}{\eta_q^2(Q_{q,n}1)} + \frac{1}{2} \sum_{j,l\in\mathbb{M}} \sum_{q=0}^n \omega_{q-1}^j \omega_{q-1}^l \frac{[\eta_q^j(f_q) - \eta_q^l(f_q)]^2}{\eta_q^2(Q_{q,n}1)}.$$

Consequently, we check that

$$W_n(f) \leq \max_{0\leq q\leq n-1} \max_{j\in\mathbb{M}} \left(\frac{\omega_q^j}{\rho_j}\right) V_n(f),$$

hence, if we are able to adapt, at each step q, the number of resampled particles N_q^j in order that $\omega_q^j \approx N_q^j/N$, we will obtain an asymptotic variance less than the asymptotic variance of the classical algorithm.

Remark 3. For any $q \geq 0$, the function $g_{q,n} := Q_{q,n} g_n = Q_{q,n+1} 1$ is also expressed as follows: for any excursion $e = (z(u), s \leq u \leq t)$ by

$$g_{q,n}(e) = g_q(e)\mathbb{P}(T_n \leq T | (T_q, Z_{T_q}) = \pi(e)).$$

Setting

$$\Delta_q^n(t,z) = \mathbb{P}(T_n \le T | T_q = t, Z_{T_q} = z),$$

and introducing the entrance distribution $\mu_q := \widehat{\eta}_q \circ \pi^{-1}$ give the following identities, since $g_q^2 = g_q$,

$$\eta_q(g_{q,n}) = \eta_q\left(g_q \Delta_q^n \circ \pi\right) = \eta_q(g_q)\widehat{\eta}_q\left(\Delta_q^n \circ \pi\right) = P_q \mu_q\left(\Delta_q^n\right),$$
$$\eta_q[g_{q,n}^2] = \eta_q\left(g_q[\Delta_q^n \circ \pi]^2\right) = \eta_q(g_q)\widehat{\eta}_q\left([\Delta_q^n \circ \pi]^2\right) = P_q \mu_q\left([\Delta_q^n]^2\right).$$

Moreover, from (2.1), we also get

$$\eta_q(g_{q,n}) = \sum_{j\in\mathbb{M}} P_q^j \mu_q^j\left(\Delta_q^n\right), \qquad \eta_q(g_{q,n}^2) = \sum_{j\in\mathbb{M}} P_q^j \mu_q^j\left([\Delta_q^n \circ \pi]^2\right),$$

with $\mu_q^j := \widehat{\eta}_q^j \circ \pi^{-1}$. We also notice that, since $\Delta_{q-1}^n \circ \pi = \mathcal{M}_q g_{q,n}$, we obtain

$$\eta_q^j(g_{q,n}) = \widehat{\eta}_{q-1}^j(\mathcal{M}_q g_{q,n}) = \mu_{q-1}^j(\Delta_{q-1}^n).$$

However, we have no so simple expression for $\eta_q^j(g_{q,n}^2)$, except perhaps by introducing the Markov kernel $R_q^{(n)}$ defined by Del Moral [5], for any function f_q acting on excursions e such that $\pi(e) \in A_{q-1}$

$$R_q^{(n)} f_q = \frac{\mathcal{M}_q(g_{q,n} f_q)}{\mathcal{M}_q g_{q,n}},$$

so that we can write

$$\eta_q^j(g_{p,n}^2) = \widehat{\eta}_{q-1}^j[\Delta_{q-1}^n \circ \pi\ R_q^{(n)} g_{q,n}] = \mu_{q-1}^j[\Delta_{q-1}^n\ R_q^{(n)} g_{q,n}].$$

As with stratified sampling [15], a lower asymptotic variance can be obtained only if the function $g_{q,n}$ is relatively homogeneous for each measure η_q^j, i.e. if $\mathrm{var}(g_{q,n}, \eta_q^j)$ is relatively small for each j and each q. This is possible if and only if we have a great variability between the values $(\eta_q^j(g_{q,n}))_{j\in\mathbb{M}}$ for each q; but the sampling per mode algorithm has been introduced precisely in order to improve the classical algorithm in that case.

4. Conclusion

In this paper we considered the rare event simulation problem for a switching diffusion, using a multilevel splitting method adapted to the discrete modes: the conditional "sampling per mode" algorithm. Using the Feynman–Kac and interacting particle theory, we established a law of large numbers and a central limit theorem for the estimate of the rare event probability and thus we confirmed that this algorithm has a better asymptotic variance as soon as the probability to hit the target set fluctuates according to the mode. We also observe that an adaptive algorithm which will update the number of resampled particles N_q^j, at each step q, in order that $\omega_q^j \approx N_q^j/N$, will give a lower asymptotic variance.

It will also be valuable to deduce from the expression of the asymptotic variance some information about the tuning of the algorithm; for example the optimal number N^j of particles in each mode. To improve again the quality of our estimate, an importance sampling technique can be added in order to make the rare switches more frequent. This idea has already been explored in [10], and now we think that it will be possible to obtain a central limit theorem also for this kind of algorithm. Let us also mention, that a possible Rao–Blackwellisation strategy could be investigated, based on a partition of the state space in the continuous component and the discrete modes.

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Appendix

The objective of this Appendix is to give a sketch of the proof of the inequality (3.12) in the context of the sampling per mode algorithm. The algorithm is defined up to the time $\tau^N = n$ the set I_n^N is empty,

$$\tau^N = \inf\{n \in \mathbb{N} \colon |I_n^N| = 0\}.$$

It follows that $\tau^N = n$ if and only if $\widehat{N}_k \neq 0$ for all $0 \leq k \leq n-1$ and $\widehat{N}_n = 0$ and that

$$\tau^N \geq n \Longleftrightarrow N_n > 0.$$

This indicate that τ^N is a predictable Markov time with respect the filtration F_n in the sense that $\{\tau^N = n\} \in F_n$ and $\{\tau^N \geq n\} \in F_{n-1}$.

Before you begin, we need to state an adapted version of the Chernov–Hoeffding inequality (see [5, Lemma 7.3.2]) for weighted empirical measure. More precisely, adopting the same notations as those used in [5], we denote by

$$m(X) = \sum_{i=1}^{N} \omega_i \delta_{X_i}$$

the N-empirical measure associated with a collection of independent random variables $X = (X_i)$ with respective distributions (μ_i). Moreover, being given N bounded functions (h_i), we use the notations

$$m(X)(h) = \sum_{i=1}^{N} \omega_i h_i(X_i), \quad \text{and} \quad \sigma^2(h) = \sum_{i=1}^{N} \omega_i^2 \mathrm{osc}^2(h_i),$$

where $\mathrm{osc}(h_i) = \sup\{|h_i(x) - h_i(y)|; x, y\}$. Now, we are ready to state the following result, the proof of which requires some minor modifications from that of [5, Lemma 7.3.2]

Lemma 6. *Suppose that $\mu_i(h_i) = 0$ for all $1 \leq i \leq N$, then*

$$\mathbb{P}(|m(X)(h)| \geq \epsilon) \leq 2e^{-2\epsilon^2/\sigma^2(h)}.$$

Theorem 7. *Suppose we have $\gamma_n(1) > 0$ for any $n \geq 0$. Then, for any N^j, $j = 1, \ldots, M$ and $n \geq 0$, we have the estimate*

$$\mathbb{P}(\tau^N \leq n) \leq a(n) e^{-N_{\inf}/c^2(n)},$$

for some constants $a(n)$ and $c(n) \leq C(n+1)/\gamma_{n+1}(1)$ which depends on n and not on N.

Proof. We proceed like in the proof of [5, Theorem 7.4.1], so we introduce the set of events $\Omega_N(n+1)$ defined by

$$\Omega_N(n+1) = \{\forall 0 \le p < q \le n+1, |\eta_p^N(Q_{p,q}1) - \eta_p(Q_{p,q}1)| \le \gamma_q(1)/2\}.$$

On the event $\Omega_N(n+1)$, we have (see [5])

$$0 < \frac{\gamma_q(1)}{2} \le \eta_p(Q_{p,q}1) - \frac{\gamma_q(1)}{2} \le \eta_p^N(Q_{p,q}1) \le \eta_p(Q_{p,q}1) + \frac{\gamma_q(1)}{2} \le 2,$$

and consequently on $\Omega_N(n+1)$, we have $\eta_p^N(g_p) \ge \gamma_{n+1}(1)/2 > 0$. This yields the inclusion $\Omega_N(n+1) \subset \{\tau^N > n\}$. On the other hand, we notice that $\Omega_N(n+1) = \Omega_N(n) \cap \Omega'_N(n)$ with

$$\Omega'_N(n) = \{\forall 0 \le p \le n, |\eta_p^N(Q_{p,n+1}1) - \eta_p(Q_{p,n+1}1)| \le \gamma_{n+1}(1)/2\}.$$

For $n = 0$, we find that $\Omega_N(1) = \{|\eta_0^N(g_0) - \eta_0(g_0)| \le \gamma_1(1)/2\}$ and by the definition of η_0^N and since the oscillation of g_0 is less that 1, using Lemma 6, we prove that

$$\mathbb{P}(\Omega_N(1)) \ge 1 - 2\exp\left(-\frac{\gamma_1^2(1)}{2\sum_{\kappa=1}^{N_0}(\beta_0^\kappa)^2}\right).$$

Furthermore, introducing the chi-square distance $\chi^2(\nu,\mu)$ of two probabilities ν and μ defined by

$$\chi^2(\nu,\mu) = \sum_i \frac{(\nu_i - \mu_i)^2}{\mu_i}$$

we obtain that

$$\sum_{\kappa=1}^{N_0}(\beta_0^\kappa)^2 = \frac{1}{N}\sum_{j=1}^{M}\frac{(\omega_0^j)^2}{\rho_j} = \frac{1}{N}(1 + \chi^2(\omega_0,\rho)),$$

where $\rho_j = N^j/N$. Hence, the inequalities

$$\mathbb{P}(\Omega_N(1)) \ge 1 - 2\exp\left(-\frac{N\gamma_1^2(1)}{2(1+\chi^2(\omega_0,\rho))}\right) \ge 1 - 2\exp\left(-\frac{N_{\inf}\gamma_1^2(1)}{2M}\right).$$

To get further, we use the decomposition

$$\mathbb{P}(\Omega_N(n+1)) = \mathbb{P}(\Omega_N(n) \cap \Omega'_N(n)) \ge \mathbb{P}(\Omega_N(n)) - \sum_{p=0}^{n}\mathbb{P}(\Omega_N(n) \cap \Omega''_N(p,n)),$$

where

$$\Omega''_N(p,n) = \{|\eta_p^N(Q_{p,n+1}1) - \eta_p(Q_{p,n+1}1)| > \gamma_{n+1}(1)/2\}.$$

Now, we can prove that [5, pp. 233–234]

$$\begin{aligned}
&\mathbb{P}(\Omega_N(n) \cap \Omega''_N(p,n)) \\
&\quad \le \sum_{k=0}^{p} \left[\mathbb{P}\left(\Omega_N(n) \cap \left\{ |\eta_k^N(Q_{k,n+1}1) - \Phi_k(\eta_{k-1}^N)(Q_{k,n+1}1)| \ge \frac{\gamma_{n+1}^2(1)}{8(n+1)} \right\} \right) \right. \\
&\qquad \left. + \mathbb{P}\left(\Omega_N(n) \cap \left\{ |\eta_k^N(Q_{k,p}1) - \Phi_k(\eta_{k-1}^N)(Q_{k,p}1)| \ge \frac{\gamma_{n+1}^2(1)}{8(n+1)} \right\} \right) \right].
\end{aligned}$$

Since, $\Omega_N(n) \subset \{\tau^N \ge n\} \subset \{\tau^N \ge k\}$ for any $0 \le k \le n$, we deduce that

$$\begin{aligned}
&\mathbb{P}\left(\Omega_N(n) \cap \left\{ |\eta_k^N(Q_{k,n+1}1) - \Phi_k(\eta_{k-1}^N)(Q_{k,n+1}1)| \ge \frac{\gamma_{n+1}^2(1)}{8(n+1)} \right\} \right) \\
&\quad \le \mathbb{P}\left(\tau^N \ge k \text{ and } \left|\eta_k^N(Q_{k,n+1}1) - \Phi_k(\eta_{k-1}^N)(Q_{k,n+1}1)\right| \ge \frac{\gamma_{n+1}^2(1)}{8(n+1)} \right) \\
&\quad \le \mathbb{E}\left(\mathbb{P}\left(\tau^N \ge k \text{ and } \left\{ |\eta_k^N(Q_{k,n+1}1) - \Phi_k(\eta_{k-1}^N)(Q_{k,n+1}1)| \right.\right.\right. \\
&\qquad \left.\left.\left. \ge \frac{\gamma_{n+1}^2(1)}{8(n+1)} \right\} | \eta_{k-1}^N \right) 1_{\{\tau^N \ge k\}} \right) \\
&\quad \le 2 \exp\left(-\frac{N_{\inf}}{32M} \left(\gamma_{n+1}^2(1)/(n+1) \right)^2 \right) \mathbb{P}(\tau^N \ge k).
\end{aligned}$$

The last displayed estimate is obtained by using Lemma 6 and the observation that $\operatorname{osc}(Q_{k,n+1}1) \le 1$.

Therefore, for each $0 \le k \le n$

$$\begin{aligned}
&\mathbb{P}\left(\Omega_N(n) \cap \left\{ |\eta_k^N(Q_{k,n+1}1) - \Phi_k(\eta_{k-1}^N)(Q_{k,n+1}1)| \ge \frac{\gamma_{n+1}^2(1)}{8(n+1)} \right\} \right) \\
&\quad \le 2 \exp\left(-\frac{N_{\inf}}{32M} \left(\gamma_{n+1}^2(1)/(n+1) \right)^2 \right).
\end{aligned}$$

Similarly, for each $0 \le k \le p \le n$

$$\begin{aligned}
&\mathbb{P}\left(\Omega_N(n) \cap \left\{ |\eta_k^N(Q_{k,p}1) - \Phi_k(\eta_{k-1}^N)(Q_{k,p}1)| \ge \frac{\gamma_{n+1}^2(1)}{8(n+1)} \right\} \right) \\
&\quad \le 2 \exp\left(-\frac{N_{\inf}}{32M} \left(\gamma_{n+1}^2(1)/(n+1) \right)^2 \right).
\end{aligned}$$

Now, using these upper bounds, we find that for any $0 \le p \le n$

$$\mathbb{P}(\Omega_N(n) \cap \Omega''_N(p,n)) \le 4(n+1) \exp\left(-\frac{N_{\inf}}{32M} \left(\gamma_{n+1}^2(1)/(n+1) \right)^2 \right)$$

and finally by recurrence

$$\begin{aligned}\mathbb{P}(\Omega_N(n+1)) &\geq \mathbb{P}(\Omega_N(n)) - 4(n+1)^2 \exp\left(-\frac{N_{\inf}}{32M}\left(\gamma_{n+1}^2(1)/(n+1)\right)^2\right)\\ &\geq \mathbb{P}(\Omega_N(1)) - 4(n+1)^3 \exp\left(-\frac{N_{\inf}}{32M}\left(\gamma_{n+1}^2(1)/(n+1)\right)^2\right)\\ &\geq 1 - 8(n+1)^3 \exp\left(-\frac{N_{\inf}}{32M}\left(\gamma_{n+1}^2(1)/(n+1)\right)^2\right).\end{aligned}$$

This ends the proof of the theorem. □

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